FP-INJECTIVE AND WEAKLY QUASI-FROBENIUS RINGS

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Abstract. The classes of FP-injective and weakly quasi-Frobenius rings are investigated. The properties for both classes of rings are closely linked with embedding of finitely presented modules in fp-flat and free modules respectively. Using these properties, we characterize the classes of coherent CF and FGF-rings. Moreover, it is proved that the group ring $R(G)$ is FP-injective (weakly quasi-Frobenius) if and only if the ring R is FP-injective (weakly quasi-Frobenius) and the group G is locally finite.

0. *Introduction*

An application of the duality context with respect to the bimodule ${}_RR_R$ to the categories of finitely generated left and right R-modules leads to the case when R is a noetherian self-injective ring. Such rings are called quasi-Frobenius (or QF-rings). In turn, an R-duality for categories of finitely presented modules leads to the class of weakly quasi-Frobenius rings (or WQF-rings). Such rings can be described as coherent FP-injective rings [**1**].

In the present paper we continue an investigation of the classes of FP-injective and WQF-rings. To begin with, one must introduce a notion of an FP-cogenerator, which plays an essential role in our analysis, aproximately the same one as the notion of the cogenerator for the class of QF-rings. Using also properties of fp-flat and fp-injective modules, we give new criteria for both classes of rings (theorems 2.2, 2.8, and 2.9), which allow to describe also the classes of coherent CF and FGF-rings. Moreover, it is proved analogs of Renault's and Connell's theorems for the FP-injective and weakly quasi-Frobenius group rings respectively (theorems 3.2 and 3.5).

It should be emphasized that the most difficult with the technical point of view statements for the FP-injective rings are proved with the help of the category of generalized R-modules ${}_R\mathcal{C} = (\mathrm{mod}-R, \mathrm{Ab})$ which consist of additive covariant functors from the category of finitely presented right R-modules $\mathrm{mod}-R$ to the category of abelian groups Ab. In our situation this is the typical case since it is localizing subcategories of the category ${}_R\mathcal{C}$ and corresponding to them torsion functors enable to adapt many properties we are interested in of the category of modules to the category of finitely presented modules. It is with the latter category the most interesting statements for FP-injective and WQF-rings are linked.

Throughout the paper the category of left (respectively right) R-modules is denoted by $R-\mathrm{Mod}$ (respectively $\mathrm{Mod}-R$) and the category of finitely presented left (respectively right) R-modules by $R-\mathrm{mod}$ (respectively $\mathrm{mod}-R$).

The dual module $\mathrm{Hom}_R(M,R)$ of $M \in R-\mathrm{Mod}$ is denoted by M^*. Regular rings are supposed to be von Neumann regular.

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1. *Preliminaries*

Recall that the *category of generalized left R-modules*

$$_R\mathcal{C} = (\mathrm{mod}\,{-R}, \mathrm{Ab})$$

consist of additive covariant functors from the category of the finitely presented right R-modules $\mathrm{mod}\,{-R}$ to the category of abelian groups Ab. In this section we give some properties of the category $_R\mathcal{C}$ used later. For more detailed information about the category $_R\mathcal{C}$ we refer the reader to [**2**] and here we, for the most part, shall adhere to this paper. All subcategories considered are supposed to be full.

We say that a subcategory $\mathcal{S}$ of an abelian category $\mathcal{C}$ is a *Serre subcategory* if for every short exact sequence

$$0 \longrightarrow X \longrightarrow Y \longrightarrow Z \longrightarrow 0$$

in $\mathcal{C}$ the object $Y \in \mathcal{S}$ if and only if $X,\ Z \in \mathcal{S}$. A Serre subcategory $\mathcal{S}$ of a Grothendieck category $\mathcal{C}$ is *localizing* if it is closed under taking direct limits. Equivalently, the inclusion functor $i : \mathcal{S} \to \mathcal{C}$ admits a right adjoint $t = t_{\mathcal{S}} : \mathcal{C} \to \mathcal{S}$ which takes every object $X \in \mathcal{C}$ to the maximal subobject $t(X)$ of X belonging to $\mathcal{S}$. The functor t one calls the *torsion functor*.

An object X of a Grothendieck category $\mathcal{C}$ is *finitely generated* if whenever there are subobjects $X_i \subseteq X$ with $i \in I$ satisfying $X = \sum_{i\in I} X_i$, then there is a finite subset $J \subset I$ such that $X = \sum_{i \in J} X_i$. The subcategory of finitely generated objects is denoted by $\mathrm{fg}\,\mathcal{C}$. A finitely generated object X is called *finitely presented* if every epimorphism $\gamma : Y \to X$ with $Y \in \mathrm{fg}\,\mathcal{C}$ has the finitely generaed kernel $\mathrm{Ker}\,\gamma$. By $\mathrm{fp}\,\mathcal{C}$ we denote the subcategory consisting of finitely presented objects. Finally, we refer to a finitely presented object $X \in \mathcal{C}$ as *coherent* if every finitely generated subobject of X is finitely presented. The corresponding subcategory of coherent objects will be denoted by $\mathrm{coh}\,\mathcal{C}$.

The category $_R\mathcal{C}$ is a locally coherent Grothendieck category, that is every object $C \in {}_R\mathcal{C}$ is a direct limit $C = \varinjlim_I C_i$ of coherent objects $C_i \in \mathrm{coh}\,{}_R\mathcal{C}$. Equivalently, the category $\mathrm{coh}\,{}_R\mathcal{C}$ is abelian. Moreover, $_R\mathcal{C}$ has enough coherent projective generators $\{(M,-)\}_{M\in \mathrm{mod}\,-R}$. Thus, every coherent object $C \in \mathrm{coh}\,{}_R\mathcal{C}$ has a projective presentation

$$(N,-) \longrightarrow (M,-) \longrightarrow C \longrightarrow 0,$$

where $M,\ N \in \mathrm{mod}\,{-R}$.

We say that $M \in {}_R\mathcal{C}$ is a $\mathrm{coh}\,{}_R\mathcal{C}$*-injective object* if $\mathrm{Ext}^1_{_R\mathcal{C}}(C,M) = 0$ for every $C \in \mathrm{coh}\,{}_R\mathcal{C}$. The fully faithful functor $-\otimes_R ? : R-\mathrm{Mod} \to {}_R\mathcal{C}$, $M \mapsto -\otimes_R M$, identifies the module category $R-\mathrm{Mod}$ with the subcategory of $\mathrm{coh}\,{}_R\mathcal{C}$-injective objects of the category $_R\mathcal{C}$. In addition, the functor

$-\otimes_R M \in \operatorname{coh}{}_R\mathcal{C}$ if and only if $M \in R-\mathrm{mod}$. Furthermore, for every $C \in \operatorname{coh}{}_R\mathcal{C}$ there is also an exact sequence

$$0 \longrightarrow C \longrightarrow -\otimes_R M \longrightarrow -\otimes_R N$$

in $\operatorname{coh}{}_R\mathcal{C}$ with $M,\ N \in R-\mathrm{mod}$.

Proposition 1.1 ([**3**, **4**]). *For a ring R the following are equivalent:*
(1) *R is left coherent;*
(2) *for every finitely presented right R-module M the left R-module $M^* = \mathrm{Hom}_R(M,R)$ is finitely presented;*
(3) *for every finitely presented right R-module M the left R-module $M^* = \mathrm{Hom}_R(M,R)$ is finitely generated;*
(4) *for every coherent object $C \in \operatorname{coh}{}_R\mathcal{C}$ the left R-module $C(R)$ is finitely presented;*
(5) *for every coherent object $C \in \operatorname{coh}{}_R\mathcal{C}$ the left R-module $C(R)$ is finitely generated.*

Recall also that a monomorphism $\mu : M \to N$ in $R-\mathrm{Mod}$ is a *pure monomorphim* if for every $K \in \mathrm{Mod}-R$ the morphism $K \otimes \mu$ is a monomorphism. Equivalently, the ${}_R\mathcal{C}$-morphism $-\otimes \mu$ is a monomorphism.

In the sequel, we use the following Serre subcategories of the category $\operatorname{coh}{}_R\mathcal{C}$:

$$\mathcal{S}^R = \{C \in \operatorname{coh}{}_R\mathcal{C} \mid C(R) = 0\}$$
$$\mathcal{S}_R = \{C \in \operatorname{coh}{}_R\mathcal{C} \mid (C, -\otimes_R R) = 0\},$$

as well as the localizing subcategories $\vec{\mathcal{S}}^R$ and $\vec{\mathcal{S}}_R$ of ${}_R\mathcal{C}$

$$\vec{\mathcal{S}}^R = \{C \in {}_R\mathcal{C} \mid C = \varinjlim C_i, C_i \in \mathcal{S}^R\}$$
$$\vec{\mathcal{S}}_R = \{C \in {}_R\mathcal{C} \mid C = \varinjlim C_i, C_i \in \mathcal{S}_R\}.$$

The corresponding $\vec{\mathcal{S}}^R$-torsion and $\vec{\mathcal{S}}_R$-torsion functors will be denoted by $t_{\mathcal{S}^R}$ and $t_{\mathcal{S}_R}$.

2. *FP-injective and weakly quasi-Frobenius rings*

A left R-module M is said to be *FP-injective* (or *absolutely pure*) if for every $F \in R-\mathrm{mod}$ we have: $\mathrm{Ext}^1_R(F,M) = 0$, or equivalently, every monomorphism $\mu : M \to N$ is pure [**5**, 2.6]. The ring R is *left FP-injective* if the module ${}_RR$ is FP-injective. M is an *fp-injective module* if for every monomorphism $\mu : K \to L$ in $R-\mathrm{mod}$ the morphism (μ, M) is an epimorphism. Clearly, FP-injective modules are fp-injective and every finitely presented fp-injective module is FP-injective. M is called *fp-flat* if for every monomorphism $\mu : K \to L$ in $\mathrm{mod}-R$ the morphism $\mu \otimes M$ is a monomorphism.

We refer to a left R-module K as an *FP-cogenerator* if for every non-zero homomorphism $f : M \to N$ from the finitely generated module M to the finitely presented module N there exists $g \in \mathrm{Hom}_R(N,K)$ such that $gf \neq 0$. K is said to be an *fp-cogenerator* if for every non-zero homomorphism $f : M \to N$ in $R-\mathrm{mod}$ there exists $g \in \mathrm{Hom}_R(N,K)$ such that $gf \neq 0$.

Obviosly, FP-cogenerators are fp-cogenerators. On the other hand, it is not hard to see that any fp-cogenerator is an FP-cogenerator when the ring R is left coherent.

Lemma 2.1. *For a left R-module K the following assertions are equivalent:*

(1) *K is an FP-cogenerator;*

(2) *every finitely presented left R-module embeds in a product $K^I = \prod_I K$ of copies of the module K;*

(3) *for every finitely presented left R-module M the following relation holds:*

$$\bigcap_{\varphi \in \mathrm{Hom}_R(M,K)} \mathrm{Ker}\,\varphi = 0.$$

Proof. (1) $\Rightarrow$ (2). Let $M \in R-\mathrm{mod}$, $I = \mathrm{Hom}_R(M,K)$, and let $\mu : M \to K^I$ be the homomorphism such that $\mu(x) = (\varphi(x))_{\varphi \in I}$. If $0 \neq x \in M$ and $i : Rx \to M$ is an inclusion, there is $\varphi : M \to K$ such that $\varphi i(x) \neq 0$. Thus, μ is a monomorphism.

(2) $\Rightarrow$ (1). Let $0 \neq f : M \to N$ be a homomorphism from the finitely generated module M to the finitely presented module N. By assumption, there exists a monomophism $g = (g_i)_{i \in I} : N \to K^I$. Then $gf \neq 0$, and so there is $i_0 \in I$ such that $g_{i_0} f \neq 0$.

(1) $\Rightarrow$ (3). Suppose $\mu : M \to K^I$ is the monomorphism constructed in the proof of the implication (1) $\Rightarrow$ (2). Then

$$\bigcap_{\varphi \in \mathrm{Hom}_R(M,K)} \mathrm{Ker}\,\varphi = \mathrm{Ker}\,\mu = 0.$$

(3) $\Rightarrow$ (2). We may take $I = \mathrm{Hom}_R(M,K)$. □

Recall that a module $M \in R-\mathrm{Mod}$ is *semireflexive* (respectively *reflexive*) if the canonical homomorphism $M \to M^{**}$ is a monomorphism (respectively isomorphism).

We are now in possession of all the information for proving the following statement (cf. [**1**, 2.5]):

Theorem 2.2. *For a ring R the following conditions are equivalent:*

(1) *the module R_R is FP-injective;*

(2) *the module ${}_RR$ is an FP-cogenerator;*

(3) *in $R - \mathrm{Mod}$ there is an fp-flat FP-cogenerator;*

(4) *in $R - \mathrm{Mod}$ there is an fp-flat cogenerator;*

(5) *if $\alpha : M \to N$ is a morphism in $R - \mathrm{mod}$ such that $\alpha^* = \mathrm{Hom}_R(\alpha, R)$ is an epimorphism, then α is a monomorphism;*

(6) *every finitely presented left R-module is semireflexive;*

(7) *every finitely presented left R-module embeds in an fp-flat module;*

(8) *every (injective) left R-module embeds in an fp-flat module;*

(9) *every FP-injective left R-module is fp-flat;*

(10) *every injective left R-module is fp-flat;*

(11) *every indecomposable injective left R-module is fp-flat;*

(12) *every flat right R-module is fp-injective.*

Proof. (1) ⇒ (2). According to [**1**, 2.5] $\mathcal{S}_R \subseteq \mathcal{S}^R$ in ${}_R\mathcal{C}$. Consider a non-zero homomorphism $f : M \to N$ with $M \in \mathrm{fg}(R-\mathrm{Mod})$ and $N \in R-\mathrm{mod}$. Suppose $C = \mathrm{Im}(-\otimes f)$; then C is a finitely generated subobject of the coherent object $-\otimes_R N$. Therefore $C \in \mathrm{coh}\,{}_R\mathcal{C}$. Assume that $gf = 0$ for every $g \in N^*$. Consider an arbitrary ${}_R\mathcal{C}$-morphism $\gamma : C \to -\otimes_R R$. Since $-\otimes_R R$ is a $\mathrm{coh}\,{}_R\mathcal{C}$-injective object, there exists $-\otimes h : -\otimes_R N \to -\otimes_R R$ such that $-\otimes h|_C = \gamma$. But $hf = 0$, and so $\gamma = 0$. Whence we obtain that $C \in \mathcal{S}_R$, and thus $C \in \mathcal{S}^R$. We see that $C(R) = 0$, which yields $f = 0$, a contradiction.

(2) ⇒ (3). By assumption, the module ${}_RR$ is an FP-cogenerator.

(3) ⇒ (7). Since a direct product of fp-flat modules is an fp-flat module (see [**1**, 2.3]), our statement follows from lemma 2.1.

(7) ⇒ (1). Let $f : M \to F$ be an embedding of a module $M \in R-\mathrm{mod}$ in an fp-flat module F. Denote by $X = \mathrm{Ker}(-\otimes f)$. Let $X = \sum_I X_i$, where each $X_i \in \mathrm{fg}\,{}_R\mathcal{C}$. Since X_i is a subobject of the coherent object $-\otimes_R M$, the object X_i is coherent itself. Because $X(R) = 0$, each $X_i \in \mathcal{S}^R$. Consequently, $X \in \vec{\mathcal{S}}^R$.

Let us apply now the left exact $\mathcal{S}_R$-torsion functor $t_{\mathcal{S}_R}$ to the exact sequence

$$0 \longrightarrow X \longrightarrow -\otimes_R M \longrightarrow -\otimes_R F.$$

Since $t_{\mathcal{S}_R}(-\otimes_R F) = 0$ (see [**1**, 2.2]), one gets that $t_{\mathcal{S}_R}(-\otimes_R M) = t_{\mathcal{S}_R}(X) \subseteq X \in \vec{\mathcal{S}}^R$. Now let $C \in \mathcal{S}_R$; since C is a subobject of $-\otimes_R M$ for some $M \in R-\mathrm{mod}$, from the relation

$$C = t_{\mathcal{S}_R}(C) \subseteq t_{\mathcal{S}_R}(-\otimes_R M) \in \vec{\mathcal{S}}^R$$

we deduce that $\mathcal{S}_R \subseteq \mathcal{S}^R$, whence the ring R is right FP-injective by [**1**, 2.5].

(1) ⇒ (5). Let $C = \mathrm{Ker}(-\otimes \alpha)$; then $C \in \mathrm{coh}\,{}_R\mathcal{C}$ and since $-\otimes_R R$ is a $\mathrm{coh}\,{}_R\mathcal{C}$-injective object, there is an exact sequence of abelian groups

$$(-\otimes_R N, -\otimes_R R) \overset{(-\otimes\alpha,-\otimes_R R)}{\longrightarrow} (-\otimes_R M, -\otimes_R R) \longrightarrow (C, -\otimes_R R) \longrightarrow 0. \tag{2.1}$$

Because α^* is an epimorphism, we conclude that $C \in \mathcal{S}_R \subseteq \mathcal{S}^R$. Thus $C(R) = 0$, and hence α is a monomorphism.

(5) ⇒ (1). Let $C \in \mathcal{S}_R$; then there is an exact sequence

$$0 \to C \to -\otimes_R M \overset{-\otimes\alpha}{\to} -\otimes_R N,$$

which induces an exact sequence of the form (2.1). Since $(C, -\otimes_R R) = 0$, α^* is an epimorphism, and hence a monomorphism. So $C \in \mathcal{S}^R$. By [**1**, 2.5] the module R_R is FP-injective.

(1) ⇔ (6). This follows from [**7**, 2.3].

(4) ⇒ (3). Obvious.

(1) ⇒ (9), (10), (11). This is a consequence of [**1**, 2.5].

(9) ⇒ (10) ⇒ (11). Straightforward.

(11) ⇒ (4). Suppose that $\{M_i \mid i \in I\}$ is a system of representatives for isomorphism classes of simple R-modules. Then every $E_i = E(M_i)$ is

an indecomposable injective R-module and the module $E = \prod_I E(M_i)$ is a cogenerator in $R-\mathrm{Mod}$. Because every module E_i is fp-flat, the module E is fp-flat by [**1**, 2.3].

(8) $\Rightarrow$ (7). Easy.

(10) $\Rightarrow$ (8). It suffices to observe that the injective hull $E(M)$ of the module M is an fp-flat module.

(1) $\Rightarrow$ (12). This is a consequence of [**1**, 2.5].

(12) $\Rightarrow$ (1). Since the module R_R is flat, it is fp-injective, and so is FP-injective. □

A left (respectively right) ideal I of the ring R is *annulet* if $I = \mathfrak{l}(X)$ (respectively $I = \mathfrak{r}(X)$), where X is some subset of the ring R and $\mathfrak{l}(X) = \{r \in R \mid rX = 0\}$ (respectively $\mathfrak{r}(X) = \{r \in R \mid Xr = 0\}$). According to [**8**] the left ideal I is annulet if and only if $I = \mathfrak{l}\mathfrak{r}(I)$.

Proposition 2.3. *For a ring R the following assertions hold:*

(1) *if R_R is an FP-injective module, then*
   (a) *for arbitrary finitely generated right ideals I, J of the ring R one has: $\mathfrak{l}(I \cap J) = \mathfrak{l}(I) + \mathfrak{l}(J)$;*
   (b) *for an arbitrary finitely generated left ideal I one has: $I = \mathfrak{l}\mathfrak{r}(I)$.*

(2) *if for R the conditions* (a) *and* (b) *from* (1) *hold, then every homomorphism of a finitely generated right ideal of the ring R into R can be extended to R.*

Proof. The proof is similar to that of [**9**, 12.4.2]. □

Corollary 2.4. *Under the conditions (a) and (b) of proposition 2.3 if the ring R is right coherent, then it is a right FP-injective ring.*

Proof. By proposition 2.3 we have $\mathrm{Ext}^1_R(R/I, R) = 0$ for every finitely generated right ideal I of the ring R. From [**5**, 3.1] we deduce that $\mathrm{Ext}^1_R(M, R) = 0$ for every $M \in \mathrm{mod}-R$, i.e. R is right FP-injective. □

Now let us consider the situation when every finitely presented left module embeds in a free module. Rings with such a property one calls *left IF-rings.* In view of theorem 2.2 any left IF-ring is right FP-injective. The next statement extends the list of properties characterizing the IF-rings (cf. [**3**]).

Proposition 2.5. *For a ring R the following conditions are equivalent:*

(1) *R is a left IF-ring;*
(2) *every finitely presented left R-module embeds in a flat R-module;*
(3) *every FP-injective left R-module is flat;*
(4) *every injective left R-module is flat.*

Proof. (1) $\Rightarrow$ (2) is trivial.

(2) $\Rightarrow$ (1). Let $f : K \to M$ be an embedding of a module $K \in R-\mathrm{mod}$ in a flat module M. By theorem of Govorov and Lazard [**8**, 11.32] the module M is a direct limit $\varinjlim_I P_i$ of the projective modules P_i. By [**6**, V.3.4] there

is $i_0 \in I$ such that f factors through P_{i_0}. Therefore K is a submodule of P_{i_0}. It remains to observe that P_{i_0} is a submodule of some free module.

(1) $\Leftrightarrow$ (4). This follows from [**3**, 2.1].

(3) $\Rightarrow$ (4). Obvious.

(4) $\Rightarrow$ (3). If M is an FP-injective left R-module, then the sequence

$$0 \longrightarrow M \longrightarrow E \longrightarrow E/M \longrightarrow 0,$$

in which $E = E(M)$, is pure. Since E is a flat module, the module E/M is flat by [**6**, I.11.1]. Let $\widehat{M} = \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$ denote the character module of M. By [**6**, I.10.5] the modules $\widehat{E}$ and $\widehat{E/M}$ are injective, and so the exact sequence

$$0 \longrightarrow \widehat{E/M} \longrightarrow \widehat{E} \longrightarrow \widehat{M} \longrightarrow 0$$

splits. Consequently, the module $\widehat{M}$ is injective, and so M is flat by [**6**, I.10.5]. □

Colby [**3**] has constructed an example of a left IF-ring, which is not a right IF-ring.

Proposition 2.6. *If R is a left FP-injective ring and a left IF-ring, then it is right coherent.*

Proof. By assumption, every $K \in R-\mathrm{mod}$ embeds in a free module (and so in a finitely generated free module as well). One has the following exact sequence

$$0 \longrightarrow K \longrightarrow R^n \longrightarrow R^m \tag{2.2}$$

in $R-$mod. Since the module ${}_RR$ is FP-injective, one gets an exact sequence

$$R^m \longrightarrow R^n \longrightarrow K^* \longrightarrow 0 \tag{2.3}$$

in $\mathrm{mod}-R$, hence $K^* \in \mathrm{mod}-R$. By proposition 1.1 the ring R is right coherent. □

IF-problem. Is it true that any left IF-ring is right coherent?

It should be remarked that IF-problem, in view of proposition 2.5, is equivalent to Jain's problem [**7**, p. 442]: will be the ring R right coherent if every injective left R-module is flat?

We recall that the ring R is *almost regular* (see [**1**]) if every (both left and right) module is fp-flat. By theorem 2.2 almost regular rings are two-sided FP-injective rings.

Corollary 2.7. *An almost regular ring R will be a left IF-ring if and only if it is regular.*

Proof. Clearly, an almost regular ring is regular if and only if it is left or right coherent. Therefore our assertion immediately follows from proposition 2.6. □

Recall also that the ring R is *indiscrete* if it is a simple almost regular ring. Prest, Rothmaler and Ziegler [**10**] have constructed an example of a non-regular indiscrete ring.

The ring R is said to be *weakly quasi-Frobenius* (or WQF-*ring*) if it determines an R-duality between the categories of finitely presented left and right R-modules. Such rings can be described as (left and right) FP-injective (left and right) coherent rings [**1**, 2.11].

The next theorem extends the list of properties characterizing the WQF-rings (cf. [**1**, 2.11; 2.12], [**3**, 2.2]).

Theorem 2.8. *For a ring R the following assertions are equivalent:*

(1) *R is a* WQF*-ring;*

(2) *R is a left and right IF-ring;*

(3) *R is left coherent and left FP-injective, and every cyclic finitely presented left R-module embeds in a free module;*

(4) *every left and every right FP-injective R-module is flat;*

(5) *every left and every right injective R-module is flat.*

Proof. (1) $\Rightarrow$ (4). This follows from [**1**, 2.11].

(2) $\Leftrightarrow$ (4) $\Leftrightarrow$ (5). Apply proposition 2.5.

(2) $\Rightarrow$ (1). Since any left and right IF-ring is a two-sided FP-injective ring, our statement follows from proposition 2.6.

(1) $\Rightarrow$ (3). Straightforward.

(3) $\Rightarrow$ (1). Since for every cyclic $K \in R-\mathrm{mod}$ the dual module $K^* \neq 0$, the proof of right FP-injectivity of the ring R is similar to that of [**1**, 2.9].

Let us show that the ring R is right coherent. In view of proposition 1.1 it suffices to prove that $K^* \in \mathrm{mod}-R$ for every $K \in R-\mathrm{mod}$. We use induction on the number of generators n of the module K. When $n = 1$, considerating exact sequences (2.2) and (2.3) for K, one gets $K^* \in \mathrm{mod}-R$. If K is finitely presented on n generators, let K' be the submodule of K generated by one of these generators. Since R is left coherent, the modules K' and K/K' are finitely presented on less than n generators. Because R is left FP-injective, one has an exact sequence

$$0 \longrightarrow (K/K')^* \longrightarrow K^* \longrightarrow (K')^* \longrightarrow 0,$$

where both $(K')^*$ and $(K/K')^*$ are finitely presented by induction. Thus $K^* \in \mathrm{mod}-R$. $\square$

Now we combine the preceding arguments in the following theorem (cf. properties of QF-rings [**11**, 24.4], [**9**, 13.2.1] and also properties of rings with full duality [**9**, 12.1.1]):

Theorem 2.9. *For a two-sided coherent ring R the following conditions are equivalent:*

(1) *R is a* WQF*-ring;*

(2) *the modules $_RR$ and R_R are FP-injective;*

(3) *$_RR$ and R_R are FP-cogenerators;*

(4) *the module R_R is an FP-injective FP-cogenerator;*

(5) *every left and every right finitely presented R-module is reflexive;*

(6) *every left and every right cyclic finitely presented R-module is reflexive;*

(7) *every left and every right cyclic finitely presented R-module embeds in a free module;*

(8) *for a finitely generated left ideal I and for a finitely generated right ideal J of the ring R one has: $\mathfrak{lr}(I) = I$ and $\mathfrak{rl}(J) = J$.*

Proof. (1) $\Leftrightarrow$ (2). This follows from [**1**, 2.11].

(2) $\Leftrightarrow$ (3) $\Leftrightarrow$ (4). Apply theorem 2.2.

(2) $\Leftrightarrow$ (5). This follows from [**5**, 4.9].

(5) $\Rightarrow$ (6). Obvious.

(6) $\Rightarrow$ (7). Let M be a cyclic finitely presented left R-module. In view of proposition 1.1 the module $M^* \in \mathrm{mod}-R$, and so there is an epimorphism $R^n \to M^*$. Hence $M \approx M^{**} \to R^n$ is a monomorphism.

(7) $\Rightarrow$ (8). Let I be a finitely generated left ideal of the ring R. By assumption, the module R/I embeds in a free module. By [**11**, 20.26] there exists a finite subset X of R such that $I = \mathfrak{l}(X)$, i. e., I is an annulet ideal. By symmetry, every finitely generated right ideal is annulet.

(8) $\Rightarrow$ (2). In view of corollary 2.4 it suffices to show that for arbitrary finitely generated right ideals I and J of the ring R the following equality holds: $\mathfrak{l}(I \cap J) = \mathfrak{l}(I) + \mathfrak{l}(J)$. Since R is coherent by assumption, by the Chase theorem [**6**, I.13.3] both $I \cap J$ and $\mathfrak{l}(I) + \mathfrak{l}(J)$ are finitely generated ideals.

One has

$$\mathfrak{rl}(I \cap J) = I \cap J = \mathfrak{rl}(I) \cap \mathfrak{rl}(J) = \mathfrak{r}(\mathfrak{l}(I) + \mathfrak{l}(J)).$$

Applying $\mathfrak{l}$, one gets

$$\mathfrak{l}(I \cap J) = \mathfrak{lr}(\mathfrak{l}(I) + \mathfrak{l}(J)) = \mathfrak{l}(I) + \mathfrak{l}(J).$$

Thus R_R is FP-injective. Likewise, ${}_RR$ is FP-injective. □

It is well-known that QF-rings have the global dimension to be equal to 0 (and then the ring R is semisimple), or ∞. In turn, WQF-rings, in view of [**5**, 3.6], have the weak global dimension to be equal to 0 (and then the ring R is regular), or ∞.

Some examples of WQF-rings the reader can find in [**3**].

Next, we consider the class of rings over which every finitely generated left R-module embeds in a free R-module. Such rings we shall call *left* FGF-*rings.* Clearly, any FGF-ring will be an IF-ring. In turn, if every cyclic left R-module embeds in a free R-module, the ring R one calls a *left* F-*ring.* The following problems are still open (see [**12**]):

FGF-problem. Does the class of left FGF-rings coincide with the class of QF-rings?

CF-problem. Will be left CF-rings left artinian?

Recall also that the ring R is a *left Kasch ring* if the injective hull $E({}_RR)$ of the module ${}_RR$ is an injective cogenerator in $R-\mathrm{Mod}$. Equivalently, for

every non-zero cyclic left R-module M the dual module $M^* \neq 0$ (see [**6**, XI.5.1]).

Lemma 2.10. *For a ring R the following assertions hold:*

(1) *if R is a left coherent ring and a left* CF*-ring, then it is a left noetherian ring and a left Kasch ring;*

(2) *if R is a left noetherian ring and a left IF-ring, then it is a left* FGF*-ring. If R is a left coherent and a left* FGF*-ring, then it is a left noetherian ring and a left IF-ring.*

Proof. (1). It is easy to see that R is a left Kasch ring. Thus we must show that the module ${}_R R$ is noetherian.

Suppose I is a left ideal of the ring R. By assumption, the module R/I is a submodule of a free R-module R^n for some $n \in \mathbb{N}$. Since the ring R is left coherent, the module R^n is coherent, and hence the module R/I is finitely presented, i.e., I is a finitely generated ideal.

(2). It is necessary to observe that over a noetherian ring every finitely generated module is finitely presented and also make use of the first statement. □

Proposition 2.11. *For a two-sided coherent ring R the following assertions are equivalent:*

(1) *R is a left* FGF*-ring;*

(2) *the module ${}_R R$ is a noetherian FP-cogenerator;*

(3) *the module ${}_R R$ is noetherian and the module R_R is FP-injective;*

(4) *R is a left noetherian ring, a left Kasch ring, and the module $E({}_R R)$ is flat.*

Proof. (1) $\Rightarrow$ (2), (3). This follows from lemma 2.10, lemma 2.1 and theorem 2.2.

(1) $\Rightarrow$ (4). Apply lemma 2.10 and proposition 2.5.

(2) $\Leftrightarrow$ (3). This is a consequence of theorem 2.2.

(2) $\Rightarrow$ (1). Since R is a left noetherian ring, every finitely generated left R-module is finitely presented. By lemma 2.1 every finitely presented left R-module is a submodule of the module R^I. Because the ring R is right coherent, the module R^I is flat by [**6**, I.13.3]. By proposition 2.5 R is a left IF-ring and by lemma 2.10 R is also a left FGF-ring.

(4) $\Rightarrow$ (1). In this case the proof is similar to the proof of the implication (2) $\Rightarrow$ (1) if we observe that every finitely generated left R-module is a submodule of the flat module E^I. □

The ring R is called *left semiartinian* if every non-zero cyclic left R-module has a non-zero socle. R is *semiregular* if $R/\operatorname{rad} R$ is a regular ring.

Lemma 2.12. [**12**, 2.1] *A left* CF*-ring R is left semiartinian if and only if* $\operatorname{soc}({}_R R)$ *is an essential submodule in ${}_R R$.*

The next two statements partially solve CF- and FGF-problems respectively (cf. [**12**]):

Proposition 2.13. *Suppose R is a left noetherian ring and a left* CF*-ring. Then the following conditions are equivalent for R:*
(1) *R is left artinian;*
(2) *R is left or right semiartinian;*
(3) *R is a semiregular ring;*
(4) *R is a semiperfect ring;*
(5) $\mathrm{soc}({}_RR)$ *is an essential submodule in ${}_RR$.*

Proof. (1) $\Rightarrow$ (2). Any left artinian ring is left perfect, and so is right semiartinian by [**6**, VIII.5.1].

(2) $\Rightarrow$ (1). Our assertion follows from [**6**, VIII.5.2].

(1) $\Rightarrow$ (3). Easy.

(3) $\Rightarrow$ (4). Since $\mathrm{soc}({}_RR)$ is a finitely generated left ideal of R, our assertion follows from [**12**, 4.1].

(4) $\Rightarrow$ (5). Obvious.

(5) $\Rightarrow$ (1). By the preceding lemma R is left semiartinian and since R is left noetherian, our assertion follows from [**6**, VIII.5.2]. □

Proposition 2.14. *Let R be a left noetherian ring and a left* FGF*-ring. Then the following are equivalent for R:*
(1) *R is a* QF*-ring;*
(2) *R is a* WQF*-ring;*
(3) *R is a left FP-injective ring;*
(4) *R is a right Kasch ring;*
(5) *R is a semiregular ring;*
(6) *R is left or right semiartinian;*
(7) $\mathrm{soc}({}_RR)$ *is an essential submodule in ${}_RR$.*

Proof. The implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3), (1) $\Rightarrow$ (4), (5) are obvious.

(3) $\Rightarrow$ (1). Our assertion follows from [**1**, 2.6].

(4) $\Rightarrow$ (3). Over a right Kasch ring the module $M^* \neq 0$ for every non-zero cyclic right R-module M. Since R is a left FGF-ring, by theorem 2.2 R is a right FP-injective ring. Therefore R is a left FP-injective ring by [**1**, 2.9].

(5) $\Leftrightarrow$ (6) $\Leftrightarrow$ (7). Apply proposition 2.13.

(7) $\Rightarrow$ (1). By proposition 2.13 the ring R is left artinian. Since any left artinian ring is right perfect, our assertion follows from [**12**, 2.5]. □

3. *Group rings*

Let R be a ring and G a group. Denote the group ring of G with coefficients in R by $R(G)$.

Lemma 3.1. *Suppose G is a finite group and $M \in \mathrm{Mod}-R(G)$. Let $f = (f_i)_I : M \to R^I$ is some R-monomorphism with I some set of indices. Then the $R(G)$-homomorphism $\widetilde{f} = (\widetilde{f_i})_I : M \to R(G)^I$ defined by the rule*

$$\widetilde{f_i}(m) = \sum\nolimits_{g\in G} f_i(mg)g^{-1}$$

is an $R(G)$-monomorphism.

Proof. It is directly verified that $\widetilde{f}$ is indeed a monomorphism of $R(G)$-modules. □

According to Renault's theorem [**13**] the group ring $R(G)$ is left self-injective if and only if the ring R is left self-injective and the group G is finite. In turn, for the FP-injective rings there is the following statement.

Theorem 3.2. *The group ring $R(G)$ is left FP-injective if and only if the ring R is left FP-injective and the group G is locally finite.*

Proof. Suppose $R(G)$ is left FP-injective and $M \in \operatorname{mod}-R$. Then the module $M \otimes_R R(G) \in \operatorname{mod}-R(G)$ and, in view of lemma 2.1 and theorem 2.2, there exists a monomorphism $\mu : M \otimes_R R(G) \to R(G)^I$ with I some set of indices. Then the composition

$$M \xrightarrow{\beta} M \otimes_R R(G) \xrightarrow{\mu} R(G)^I$$

of morphisms μ and β is an R-monomorphism, where $\beta(m) = m \otimes e$, e is a unit of the group G. Since $R(G)$ is a free R-module, the R-module $R(G)^I$ is fp-flat by [**1**, 2.3]. Theorem 2.2 implies that the ring R is left FP-injective.

Let us show now that the group G is locally finite. Let H be a non-trivial subgroup of the group G generated by elements $\{h_i\}_{i=1}^n$. Then the right ideal ωH of the ring $R(G)$ generated by the elements $\{1 - h_i\}_{i=1}^n$ is non-zero. By proposition 2.3 $\mathfrak{l}(\omega H) \neq 0$, and so H is finite by [**14**, 2.1].

Now let R be left FP-injective and the group G locally finite. To begin, let us show that the ring $R(G)$ is left FP-injective if G is finite.

Suppose $M \in \operatorname{mod}-R(G)$. Because the group G is finite, $M \in \operatorname{mod}-R$. Since the module R_R is an FP-cogenerator by theorem 2.2, by lemma 2.1 M_R is a submodule of R^I, where I is some set. By lemma 3.1 $M_{R(G)}$ is a submodule of $R(G)^I$. Consequently, $R(G)$ is an FP-cogenerator, and hence $R(G)$ is a left FP-injective ring by theorem 2.2.

Next, suppose that G is an arbitrary locally finite group and $M \in R(G)-\operatorname{mod}$. Then there is a short exact sequence of $R(G)$-modules

$$0 \longrightarrow K \xrightarrow{i} R(G)^n \xrightarrow{p} M \longrightarrow 0.$$

Let X be a finite set of generators for ${}_{R(G)}K$. Because G is locally finite, there is a finite subgroup H of G such that $R(H)X \subseteq R(H)^n \subseteq R(G)^n$. We result in the short exact sequence of $R(H)$-modules

$$0 \longrightarrow K' \xrightarrow{\bar{i}} R(H)^n \xrightarrow{\bar{p}} M' \longrightarrow 0, \tag{3.1}$$

where $K' = R(H)X$, $M' = R(H)Y$, Y is a finite set of generators of the module ${}_{R(G)}M$. Tensoring sequence (3.1) on $R(G)_{R(H)}$, one gets the following

commutative diagram with exact rows:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & R(G) \bigotimes\limits_{R(H)} K' & \xrightarrow{1\otimes\bar{i}} & R(G) \bigotimes\limits_{R(H)} R(H)^n & \xrightarrow{1\otimes\bar{p}} & R(G) \bigotimes\limits_{R(H)} M' & \longrightarrow & 0 \\
 & & \Big\downarrow \alpha & & \Big\downarrow \beta & & \Big\downarrow \gamma & & \\
0 & \longrightarrow & K & \xrightarrow[i]{} & R(G)^n & \xrightarrow[p]{} & M & \longrightarrow & 0,
\end{array}$$

in which β is an isomorphism, $\alpha(r \otimes x) = rx$. Clearly, α is an isomorphism, and hence γ is an isomorphism.

If we showed that every $f \in \mathrm{Hom}_{R(G)}(K, R(G))$ is extended to some $\sigma \in \mathrm{Hom}_{R(G)}(R(G)^n, R(G))$, we would obtain that $\mathrm{Ext}^1_{R(G)}(M, R(G)) = 0$, as required.

So suppose that $f \in \mathrm{Hom}_{R(G)}(K, R(G))$ and $\tau \in \mathrm{Hom}_{R(H)}(R(G), R(H))$, $\tau(\sum_{g\in G} r(g)g) = \sum_{g\in H} r(g)g$. Consider $\bar{f} = \tau f|_{K'} \in \mathrm{Hom}_{R(H)}(K', R(H))$. Because $R(H)$ is a left FP-injective ring, there is $\bar{\sigma} : R(H)^n \to R(H)$ such that $\bar{f} = \bar{\sigma}\bar{i}$. One gets

$$f = (1 \otimes \bar{f})\alpha^{-1} = (1 \otimes \bar{\sigma})(1 \otimes \bar{i})\alpha^{-1} = (1 \otimes \bar{\sigma})\beta^{-1} i = \sigma i,$$

where $\sigma = (1 \otimes \bar{\sigma})\beta^{-1}$ is the required homomorphism. □

The Renault theorem and theorem 3.2 implies that given an arbitrary self-injective ring R, one can construct FP-injective rings which are not self-injective. To be definite, the following statement holds:

Corollary 3.3. *If R is a left self-injective ring, G is a locally finite group, and $|G| = \infty$, then the group ring $R(G)$ is left FP-injective but not left self-injective.*

Proposition 3.4 (Colby [**3**])**.** *The group ring $R(G)$ is a left IF-ring if and only if R is a left IF-ring and the group G is locally finite.*

Proof. If $R(G)$ is a left IF-ring, it is also right FP-injective by theorem 2.2. By the preceding theorem the group G is locally finite. Similar to the proof of theorem 3.2, given $M \in R-$mod there is a composition of R-monomorphisms

$$M \xrightarrow{\beta} R(G) \otimes_R M \xrightarrow{\mu} R(G)^n$$

with $\beta(m) = e \otimes m$. Since $R(G)$ is a free R-module, the module M is a submodule of the free R-module $R(G)^n$.

Conversely, let R be a left IF-ring and G a locally finite group. First, let us prove that $R(G)$ is a left IF-ring if G is a finite group. For this consider $M \in R(G)-$mod. Since G is finite, $M \in R-$mod, and so ${}_R M$ is a submodule of R^n for some $n \in \mathbb{N}$. By lemma 3.1 ${}_{R(G)}M$ is a submodule of $R(G)^n$, i.., $R(G)$ is indeed a left IF-ring.

If G is an arbitrary locally finite group, then for any $M \in R(G) -$ mod there is a finite subgroup H of the group G such that

$$M \approx R(G) \otimes_{R(H)} R(H)Y$$

with Y a finite set of generators for ${}_{R(G)}M$ (see the proof of theorem 3.2). By assumption, the $R(H)$-module $R(H)Y$ is a submodule of a free module $R(H)^n$ for some $n \in \mathbb{N}$. Thus, M is a submodule of the free module $R(G)^n \approx R(G) \otimes_{R(H)} R(H)^n$. □

Theorem 3.5. *The group ring $R(G)$ is weakly quasi-Frobenius if and only if the ring R is weakly quasi-Frobenius and the group G is locally finite.*

Proof. Theorem 2.8 implies that any WQF-ring is a left and right IF-ring. Therefore our statement immediately follows from proposition 3.4. □

It is well-known that the group ring $R(G)$ is semisimple (see [**14, 8**]) if and only if the ring R is semisimple, the group G is finite, and $|G|$ is invertible in R. In turn, by theorem of Auslander and McLaughlin (see [**14**]) $R(G)$ is a regular ring if and only if the ring R is regular, the group G is locally finite, and for every finite subgroup H of G the equality $|H| = n$ implies $nR = R$.

To conclude, we give some examples of WQF-rings which are simultaneously neither QF-rings, nor regular rings.

Examples. (1) Given an arbitrary regular ring R, we can construct WQF-rings which will not be regular. Namely, it is necessary to consider an arbitrary locally finite group G, in which there is at least one finite subgroup H of G such that the order $|H|$ is not a unit in R.

To take an example, consider the field K of the characteristic $p \neq 0$. Let $R = \prod_{i=1}^{\infty} K_i$, $K_i = K$, be the ring with component-wise operations. Then R is a regular but not semisimple ring, as one easily sees. If G is a finite group such that p devides $|G|$, then the ring $R(G)$ is a weakly quasi-Frobenius ring being neither quasi-Frobenius, nor regular.

(2) Let R be an arbitrary QF-ring, G an arbitrary locally finite group, and $|G| = \infty$. Then $R(G)$ is a weakly quasi-Frobenius ring but not quasi-Frobenius. Moreover, $R(G)$ is regular if and only if R is a semisimple ring and the order of every finite subgroup of G is invertible in R.

As an example, if K is the field of the characteristic $p \neq 0$, the group $G = \cup_{k \geq 1} G_k$, where every G_k is a cyclic group with a generator a_k of the order p^k, and $a_k = a_{k+1}^p$, then the group algebra $K(G) = \varinjlim_k K(G_k)$ is weakly quasi-Frobenius (see also [**15**]) being neither quasi-Frobenius, nor regular.

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